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A Framework to Gauge Mathematical Understanding: A Case Study on Linear Algebra Concepts

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ABSTRACT A framework is arrived at to help gauge the level of mathematical understanding that could be suggested by an examination of a student's written responses to mathematical problems. This framework has been to make conclusions on two students' level of mathematical understanding and use of mathematical symbolism to communicate their understanding of operations on matrices in an external form to others. It was found that for those two students mathematical symbolism served largely as an instrumental role to keep track of their thinking. At times the incorrect use of a mathematical symbol, for example the equal to sign, led the student to make illogical conclusions. Further it was found that drawing students' attention to their incorrect use of mathematical symbolism and the lack of explicit communication of their thinking to others could help to clarify and even modify existing schema that they use to find possible solutions to given tasks.

INTRODUCTION

This study was informed by the researcher's interaction with first year university mathematics students. Most of those students did not pay enough attention to details in their writing of solutions to mathematics problems. Generally the presentation of solutions comprised of unrelated steps that made it difficult to follow the level of understanding of a student with regard to the relevant concept(s). The researcher was tasked to lecture the linear algebra component of a module to a group of 185 students who completed the requirements to take that module. It was decided to conduct a study on the level of mathematical understanding of students by examining their written responses to questions on some basic linear algebra concepts. This required the formulation of a framework to focus on mathematical understanding. The work in this paper differs from the author's previous work, for example Maharaj (2013, 2014, 2015) in the following ways. In this paper APOS (action-process-object-schema) Theory concepts are related to instrumental and relational understanding. The framework arrived at identifies which of the APOS concepts could possibly be linked to instrumental and relational understanding. Further which of the APOS concepts could come into play when mathematical symbolism serves an instrumental role or a communicative function is also focused on in the framework.

Research Question

The main research question was: *What does an examining of the written responses of students to linear algebra problems reveal about their level of mathematical understanding?* To answer this question the following sub-questions had to be answered: *(1) What is meant by mathematical understanding? (2) What type of framework could be used to examine the written responses of students with the focus on their level of mathematical understanding?*

Literature Review and Reflections

This focuses on: (1) What is mathematics? (2) Mathematical understanding. (3) Studies on linear algebra concepts.

What is Mathematics?

The view given by Godino (1996), based on the following assumptions, was found to be useful: a) Mathematics is a human activity involving the *solution of problematic situations*. In finding the responses or solutions to these external and internal problems, mathematics progressively emerges and evolves. b) Mathematical problems and their solutions are shared in *specific institutions or collectives* involved in studying such problems. c) Mathematics is a *symbolic language* in which problem-situations

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and the solutions found are expressed. This symbolic language, which codes information using mathematical symbols, serves an *instrumental role* (to keep track of thinking) and also a *communicative function* (to communicate thinking externally to others). d) Mathematics is a *logically organized conceptual system*. Once a mathematical concept has been accepted as a part of this system, it can also be considered as a textual reality and a component of the global structure. It may be handled as a whole to create new mathematics, widening the range of mathematical tools and, at the same time, introducing new restrictions in mathematical work and language. This is also what Menary (2015) argued when stating that mathematical cognition is an example of the process of enculturation at work. That researcher also concluded that the process of enculturation to mathematical concepts and the symbolic language used is an important component towards understanding in mathematics.

It should be noted these assumptions indicate that for the purpose of teaching certain institutional or global conventions need to be followed by, both those who teach and those who want to learn mathematics. These points imply the following in the context of teaching and learning mathematics. An integral part of the teaching and learning of mathematics should focus on the language of mathematical symbolism which is used to unpack the problem-situations (an exercise or word problem or practical real world context problem) and then present formal solutions to them. In the unpacking and solving of a problem, the use of mathematical symbolism serves two purposes, a *communicative function* and instrumental role. The latter could be viewed as a vehicle or instrument to represent, aid, keep track of or summarise internal mental thinking. Note that solving a problem and writing out a solution with the intention of communicating this to others are different skills. Where the form of examination is external in the sense of a format that is a written one and this is largely the case at university level, the skill of writing out a solution is also very important. This is where the communicative function of mathematical symbolism comes to the fore. The solution could be examined to gauge the person's level of understanding of the relevant mathematics' section under focus. A person has to demonstrate the level of his or her thinking which will include a demonstration of his or her understanding in a format that is external. What this means is that the (written) solution to a mathematics problem should have a clear thread in which the assumption and implications are clear. To indicate the logic or thinking involved explanations should be given and use should be made of connectives to link symbolic representations of mathematical concepts, for example $f(x)=\frac{y+1}{x-1}$, with mathematical symbols such as '$\therefore$' (therefore), '$\Longrightarrow$' (implies) and '$\Longleftrightarrow$' (implies and is implied by). These imply that lecturers need to be able to write out solutions that serve a communicative function. Further, the instrumental role of symbolic language as a vehicle for thinking and thought patterns should emerge from such solutions. Each of these has to be taught to students. In the researcher's opinion this is widely lacking in the type of teaching that goes on in our secondary schools. The researcher made this conclusion from his observations of and interactions with first year mathematics university students over the past two decades. It is a given that for lecturers to teach the dual purpose of mathematical symbolism, with regard to serving a communicative function and instrumental role, lecturers themselves need to have a good understanding of this. A lecturer's use of the symbolic language and demonstration to his or her students should lead them to model the lecturer's use and demonstration of this language in the context of mathematical thinking and the formal communication of such thinking.

Mathematical Understanding

What is meant by understanding in mathematics? Various educationists have written on understanding in general (Skemp 1976; Nickerson 1985; Sierpinska 1994), understanding in mathematics (Hiebert and Carpenter 1992) and how to access understanding in mathematics (Barmby et al. 2007). For example, Skemp (1976) identified two types of understanding: (a) *relational understanding* which he described as knowing what to do and why, and (b) *instrumental understanding* which he described as rules without understanding. He noted that the process of learning relational mathematics leads to the building of a conceptual structure in mathematics. It is the researcher's opinion that the focus in the teaching and learning of mathematics should be on relational understanding. In his examination of understanding Nickerson

(1985) identified the *results* of understanding, for example agreement with experts, seeing deeper characteristics of a concept, looking for specific information in a situation more quickly, ability to represent situations, or envisioning a situation using mental models. He also highlighted the importance of knowledge and relating knowledge, since he argued that understanding is enhanced by one's ability to build bridges or connections between one conceptual domain and another. So he concurred with the importance of relational understanding as described by Skemp. In particular with regard to understanding in mathematics Hiebert and Carpenter (1992) and Barmby et al. (2007) viewed this as the building of structures or context within the following framework: (1) mathematics is understood if its mental representation is connected to a network of representations; (2) the degree of this understanding is determined by the number and strength of such connections and (3) a new mathematical idea, for example a definition or procedure, is likely to be thoroughly understood if it is linked to an existing network which has numerous connections. It is these connections that are considered to facilitate the transfer of prior knowledge to novel situations. This transfer is essential since previously learned strategies are used to solve many new problems. Without such transfers each new problem will require a separate strategy (Stylianides and Stylianides 2007) and it would be impossible for one to become mathematically competent.

In their discussion of ways to access mathematical understanding Barmby et al. (2007) indicated that there is a drawback to any potential method that uses external representations of mathematical concepts to try and access connections made between internal representations. The reason for this is that there is no guarantee that the external representations will be a true reflection of what goes on in the mind of a person, including between the internal representations. Bearing this in mind the work of Hiebert and Carpenter (1992) and Barmby et al. (2007) imply that the following could serve as a useful starting point to examine the possibilities for assessing mathematical understanding: (1) students' errors; (2) *connections* made between *symbols* and *symbolic procedures* and *corresponding referents*; (3) *connections* between *symbolic procedures* and *problem solving situations* and (4) *connections* made between *different symbol systems*. Veloo et al. (2015) reported that errors made by students stemmed from: a lack of understanding; procedures being forgotten; negligence in transcribing information from the question; carelessness and guesswork. Those researchers suggested that enough work and examples should be focused on, directly addressing the identified perceived misconceptions. They argued that as misconceptions are reduced the clarity of concepts increases.

Studies on Linear Algebra Concepts

Li (2013) noted that linear algebra is one of the most important mathematical courses for undergraduate students. That researcher suggested that an over emphasization on memorization of definitions and equations found in text books always weakens students' innovation ability cultivation. Among the proposals made to address this problem are the following approaches: combining the lectures with practical problems; depicting one concept or showing problem solving illustrations from different viewpoints; providing students with opportunities to find solutions by using scientific computation software, for example Mathlab. The study by Cooley et al. (2007) showed that participants gained a deeper understanding of linear algebra by: making sense of written material; discussing ideas with others (students, instructor); reflecting on and describing their own thought processes. That study also noted the role of visualisation in helping participants to make multiple representations and making connections between those representations. These imply that encouraging students to engage with and reflect on their written responses could lead them to pathways for fostering deeper mathematical understanding.

With regard to concepts that are the essence and foundation of a linear algebra course, Stewart and Thomas (2009) noted that many of their student-participants had major difficulties with: understanding such concepts; connecting a concept related to another concept. They further noted that the majority of those students functioned at an action/process level of understanding. Action and process levels of understanding will be explained in the next section. In the current study the relevant linear algebra concepts focused on: operations with matrices; matrix multiplication. The study by Ulus (2013) focused on the effectiveness of using advanced calculators to support understanding of the diagonalization concept in linear algebra. That study concluded that advanced calculators could be used to expose students to carefully designed tasks which in turn led to the acquisition of mathematical knowledge. In the present study the module requirement was that students were not allowed to use calculators. However, the study process they were exposed to incorporated the use of carefully designed tasks for linear algebra. Those tasks were made available to students on the module website so that they could come prepared for the formal lectures and tutorials. The study by Voskoglou and Buckley (2012) sheds some light on the relationship between computational and critical thinking. Problem solving promotes critical thinking. Those researchers found that there was a strong indication that the use of computers as a tool for problem solving enhanced students' abilities to solve problems that involved mathematical modelling. In the current study both the lectures and tutorials the students were exposed to focused on problem solving and the material was available to students online in an electronic format.

The above informed the conceptual framework and methodology for this study.

Conceptual Framework

In this section the researcher focuses on: APOS (action-process-object-schema) mental structures; towards a framework for externally examining understanding.

APOS Mental Structures

The mathematician and mathematics educationist Dubinsky proposed APOS (action-process-object-schema) theory to determine the type of mental constructions required to engage with a mathematical concept. In the context of a given mathematical concept APOS theory postulates the cognitive structures required to construct knowledge through action, process, object and schema. Lately some South African researchers focused on mental constructions as proposed by APOS theory to promote the teaching and learning of particular mathematics concepts (for example Brijlall and Ndlovu 2013; Maharaj 2013, 2014). For an outline of APOS Theory and its application in the context of different mathematical concepts the reader is referred to Maharaj (2010, 2013, 2014) and Brijlall and Ndlovu (2013). In this paper it was intention of the researcher to use mental constructions in APOS theory together with the above documentations of what he believes understanding and in particular mathematical understanding to be to come up with a framework to examine and comment on the quality of external written responses of students. These would be to analyse their written responses to the type of mathematical concepts and problems that students are expected to study. So the researcher does not give a detailed account of APOS Theory but explains the concepts of action, process, object and schema. The descriptions of action, process, object and schema below are based on those given by Weller et al. (2009), Maharaj (2010, 2013, 2014) and Arnon et al. (2014).

Action: A transformation is first conceived as an *action*, when it is a reaction to stimuli which an individual perceives as external. It requires specific instructions, and *the need to perform each step of the transformation explicitly*.

Process: As an individual repeats and reflects on an action, it may be *interiorized* into a mental *process*. A process is a mental structure that performs the same operation as the action, but *wholly in the mind of the individual*. Specifically, the individual can imagine performing the transformation without having to execute each step explicitly.

Object: If one becomes aware of a process as a totality, realizes that transformations can act on that totality and can actually construct such transformations (explicitly or in one's imagination), then we say the individual has *encapsulated* the process into a cognitive *object*.

Schema: A mathematical topic often involves many *actions, processes* and *objects* that need to be *organized and linked into a coherent framework*, called a *schema*. It is coherent in that it provides an individual with a way of deciding, when presented with a particular mathematical situation, whether the schema applies.

We have no way of knowing exactly what goes on in a person's mind. The APOS mental structures could provide a means of examining different levels of understanding with regard to a particular concept. It seems that in a problem solving context these have to be managed by the problem solver. Some researchers (Kulze 2013; Carlson and Bloom 2005; Lawson and

Chinnappan 2000; Schoenfeld 1992) looked at what leads to success in problem solving. Their research demonstrated that success in problem solving performance depends on a problem solver's ability to: (1) retrieve more knowledge, (2) activate links among knowledge schemata and related information and (3) coordinate these at the same time. What is important is the management of different mathematical resources (including factual and procedural knowledge) since problem solving requires, in addition to knowing what and when to monitor, knowing how to monitor (Lester 1994).

Towards a Framework for Examining Understanding

If one wants to externally examine the depth or level of mathematical understanding of a person then it follows that the person is required to demonstrate his or her understanding. This means that the person should be able to represent his or her thinking towards a problem situation externally. The examination will be an evaluation of the quality of the person's written representation of his or her solution (and possible understanding) of a problem-situation. The above is the context in which this researcher formulated a framework to evaluate and comment on the quality of students' written responses to and possible understanding of problem situations, they were expected to study. This framework is illustrated in Table 1, which indicates the literature sources, the type of mathematical understanding focused on and relationship with the APOS mental structures. A starting point is to note that an examination of mathematical understanding should focus on whether that understanding is instrumental or relational. For example, instrumental understanding is associated with a mental structure that would be developed to at most an action level. Mathematical understanding could also be deduced from a person's ability to use mathematical symbolism, this could serve an instrumental role or communicative function. These depend on the (internal) interactions among APOS mental structures and also execution of those mental structures (for which there is external evidence). Further, the relational understanding and the use of mathematical symbolism would be overseen or guided by a logically organized conceptual system. In the context of APOS mental structures this would depend on the degree of development and interrelationships of relevant schema.

METHODOLOGY

The researcher lectured the linear algebra component to a group of 185 students. It was emphasized, at the outset, that they had to pay attention to the details in their presentation of written responses to problems. On the general information sheet for the module a table indicated the use of incorrect notation in mathematics, why it was incorrect and what the correct notation was. For example it was indicated why the following were incorrect: (1) $f(x)=x^2=2x$, (2) $g(x)=x(x+1)=x2+x$. During the lectures for linear algebra the use of the 'equal' and 'implies' signs were explained in the context of different situations. For example those were explained in the context of adding two matrices of the same size. The importance of writing down explanations so that others could follow the reasoning was also emphasized during lectures. Such details were also looked at and focused on during tutorial sessions when students presented their

Table 1: Framework to evaluate level of students' understanding

Literature source and depth	*Mathematical understanding*	*APOS mental structures*
Skemp Nickerson, Hiebert and Carpenter, Dubinsky, Maharaj	*Instrumental* Relational • Connected to an existing network • Number and strength of connections	*At most at an action level* • Process, object, schema • Connections between existing schema
Godino Hiebert and Carpenter Barmby et al. Dubinsky, Maharaj	Mathematical symbolism • Instrumental role • Communicative function actions, processes, objects, schema Logically organized conceptual system	 • Interactions among mental structures of actions, processes, objects, schema • Execution of mental structures Degree of development and interrelationships of schema

written solutions to problems. During one such tutorial session for a tutorial group in which there were 20 students, the written work of two students was first examined and then those students were interviewed. The examination of students' written work led to the formulation of primary interview questions with each student. Interview questions were formulated to get a deeper insight into the written work of each student. During the interviews further questions were formulated to probe the initial responses of each student.

The choice of the two students was random and their consent to carry out the research was obtained. Those students felt that the approach would benefit them and requested for the interviews to be continued outside the tutorial venue. During their free time the interviews were continued in the researcher's office, within a week of the tutorial interviews. Since this was a quantitative study it was decided to use the written work and interview responses of those two students. The written extracts and interview responses of those two students are referred to by S1 and S2, in the next section. The findings on those were based on the framework illustrated in Table 1. Generally an idea is better understood if an example is used to illustrate that idea. The use of only two participants to generate the evidence could be viewed as a case study. Cohen et al. (2007) argued that a case study could be chosen to illustrate a general principle.

FINDINGS AND DISCUSSION

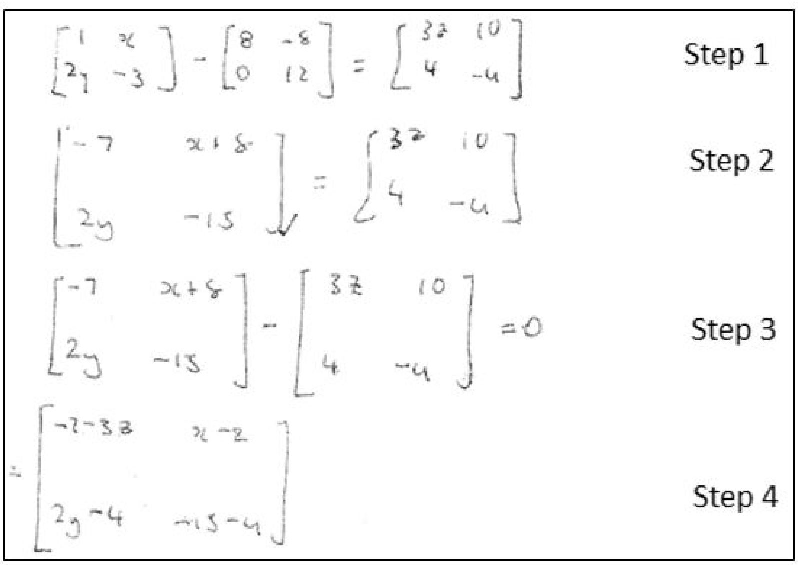
To present this in a reader friendly manner the relevant findings and discussions are indicated under the following sub-headings: Question 1 and Question 2. In each case the question that the student responded to is stated and extracts of student(s) responses are given. The latter are discussed together with relevant student and researcher responses during the interviews.

Question 1 [Operations with Matrices]

Solve for *u*, *x*, *y* and *z* in the matrix equation:

$$\begin{bmatrix} 1 & x \\ 2y & -3 \end{bmatrix} - 4\begin{bmatrix} 2 & -2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 3z & 10 \\ 4 & -u \end{bmatrix}$$

Extract 1 gives the written response of student S1 to Question 1. The matrix equation in Step 1 indicates that the student understood the concept of scalar multiplication of a matrix. She correctly multiplied each entry of the second matrix in the question by the scalar 4. The matrix equation in Step 2 indicates that the student correctly subtracted the two matrices on the left hand side of the equation in Step 1. Those two steps indicate that the student displayed instrumental understanding and the mental structures were at an action level. Having arrived at



$$\begin{bmatrix} 1 & x \\ 2y & -3 \end{bmatrix} - \begin{bmatrix} 8 & -8 \\ 0 & 12 \end{bmatrix} = \begin{bmatrix} 3z & 10 \\ 4 & -u \end{bmatrix} \quad \text{Step 1}$$

$$\begin{bmatrix} -7 & x+8 \\ 2y & -15 \end{bmatrix} = \begin{bmatrix} 3z & 10 \\ 4 & -u \end{bmatrix} \quad \text{Step 2}$$

$$\begin{bmatrix} -7 & x+8 \\ 2y & -15 \end{bmatrix} - \begin{bmatrix} 3z & 10 \\ 4 & -u \end{bmatrix} = 0 \quad \text{Step 3}$$

$$= \begin{bmatrix} -7-3z & x-2 \\ 2y-4 & -15-u \end{bmatrix} \quad \text{Step 4}$$

Extract 1: Written response of student S1

the Step 2 which represents an object in symbolic form, in this case the equality of the two matrices, the student failed to display relational understanding. Step 3 is correct provided the 0 on the right hand side is interpreted as the zero matrix in the context of that matrix equation. It is evident from the student's response that such an interpretation was not in place. Steps 3 and 4 provide further evidence that the student only had at most an instrumental understanding of scalar multiplication of a matrix and the subtraction of two matrices. In the context of the given question using the framework in Table 1, Step 4 provides evidence the student was unable to function at even an action level when confronted with the equality of two matrices for the symbolic form indicated in Step 2. The understanding displayed for the concept equality of two matrices is, therefore, not even at an instrumental level. Step 3 is true but irrelevant. Observe that Extract 1 indicates that the student did not know how to interpret Step 3. This is supported by what transpired during the interview.

R: Do you know what is incorrect here? [*pointing at Step 4*]

S1: Equal to here [*pointing in Step 3*] is wrong

R: Why?

S1: It implies that these matrices here [*pointing to LHS*] is equal to zero.

R: What does that mean?

S1: Each side is zero. I think that's what it implies …… From here [*pointing to Step 4*] I didn't know how to get the solutions for *x, y* and *z.*

This confirms that having arrived at Step 2 the mental constructions of the student were not even at an action level for the concept of equality of matrices, hence her inability to proceed correctly. To solve for the four unknown *u, x, y,* and *z* after arriving at Step 2 an interpretation of the mathematical symbolism that the two matrices are equivalent, is required. It was already noted that the student lacked the relational understanding required to proceed towards a solution. Since the process of learning relational mathematics leads to the building of a structure in mathematics (Skemp 1976) it is not surprising that student S1 was unable to arrive at a solution for Question 1. In particular here the relational mathematics is in the context of mathematical symbolism related to the linear algebra concept of equality of two matrices and its implications. These are that the sizes of the matrices must be the same and the corresponding entries must be equal. This supports the argument by Menary (2015) that mathematical cognition is an example of the process of enculturation at work. The process of enculturation here is in the context of *mathematical concepts* (for example equivalent matrices) and the *symbolic language* used (for example the interpretation of the given matrix equation). Both the mathematical concepts and the use of symbolic language to communicate relevant concepts are important pillars towards understanding in mathematics.

Extract 2 gives the written response of student S2 to Question 1. Using the framework in Table 1, this response indicates that the student's mental constructions for the solving of the given matrix equation was at least at the process level, which implies that the student had relational understanding for this concept. This is so because her written response seems to be connected to a network for solving matrix equations. The student arrived at each of the four basic equations, for example 1-8 = 3z and 2y-0=4. Those four equations provide clear evidence that for the operations of scalar multiplication of a matrix and subtraction of matrices the mental constructions were at least at a process level. The latter is true since the student clearly performed those mentally. Further the response also indicates that the student had some sort of a schema for dealing with matrix equations of the type in Question 1. Evidence of this is also given by what transpired during the interview with student S2.

When student S2 was probed on how she arrived at the written response in Extract 2, the following transpired:

R: How did you get this? [*pointing to the four basic equations in Extract 2*]

S2: I multiplied these two matrices [*pointing to the LHS in Question 1*].

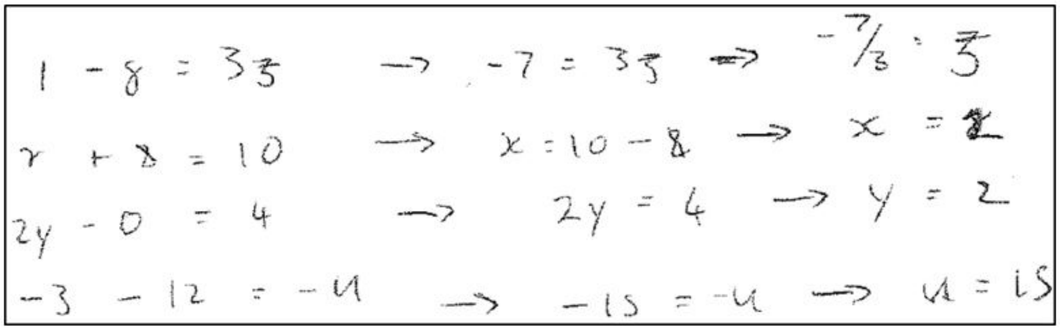
R: Explain

S2: I multiplied by the scalar 4 and subtracted the 2 matrices … these solved for the values of *u, x, y* and *z.*

R: Why did you equate 1-8 to 3z=7

S2: The position of the entry 1-8 is equal to the position of the answer 3z.

These suggest that the student had a schema for dealing with solving of matrix equations of the type indicated in Question 1. The written response and the last verbal response suggest that the student had mental structures for such


$1 - 8 = 3z \rightarrow -7 = 3z \Rightarrow -7/3 = z$

$x + 8 = 10 \rightarrow x = 10 - 8 \rightarrow x = 2$

$2y - 0 = 4 \rightarrow 2y = 4 \rightarrow y = 2$

$-3 - 12 = -u \rightarrow -15 = -u \rightarrow u = 15$

Extract 2: Written response of student S2

equations at the object level. Further, the students' use of mathematical symbolism was developed to serve at least an instrumental role. Lacking to an extent was the use of mathematical symbolism developed to the stage of a communicative function. This implies that the student was unable to correctly execute all available mental structures across different contexts in mathematics. Further evidence for these occurred during the interview. The student was questioned on her use of the symbol "$\rightarrow$" (see Extract 2).

R: What does this mean? [*pointing to the symbol* "$\rightarrow$"]

S2: Wrong notation for implies.

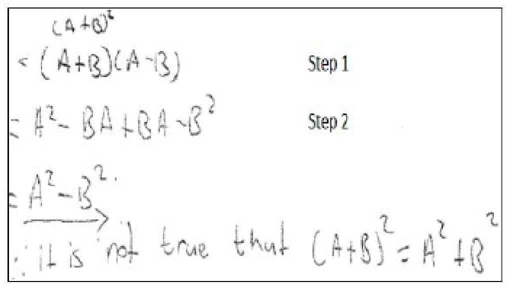
After further probing it seemed that the student confused the use of the 'approaches' symbol in the context of limits with the 'implies' symbol used in other contexts. Once this student's misconception on the symbolic representation of the 'implies' symbol was addressed the clarity of concepts related to the use of this symbol increased, which supports the suggestion by Veloo et al. (2015).

Question 2 [Matrix Multiplication is not Communicative]

If A and B are matrices with AB=0 , is it true that Explain.

$(A+B)^2 = A2+B^2$?

An examination of Extract 3 reveals two serious errors. The first appears as Step 1. Even in the context of basic algebra $(a+b)^2$ is not equal to (a+b) (a-b). This implies that the student was unable to activate links among knowledge schemata and related information. In this case the knowledge schemata for algebra that is equal to (a+b) could not be retrieved correctly and related to the context, $(a+b)^2$ where A and B represent matrices. The latter was important for the multiplication of the matrices and AB to be defined. During the interview the student's first error was probed as follows:


$(A+B)^2$

$= (A+B)(A-B)$ Step 1

$= A^2 - BA + BA - B^2$ Step 2

$= A^2 - B^2.$

$\therefore$ it is not true that $(A+B)^2 = A^2 + B^2$

Extract 3: Written response of student S1

R: Is this correct? [*pointing at (A-B)) in Step1*]

S1: It supposed to be plus here. [*pointing to the* "$\rightarrow$" *sign*]

The second serious error appears in Step 2, in particular the second term which is indicated as, –BA. It seems the student was able to correctly retrieve the knowledge in the context of algebra to expand from Step 1 an object of the form (a+b) (b-a). However, the student did not activate links among related information, in the context of matrices. Matrix multiplication is not communitative, so the second term in Step 2 should be,–AB. This was the crux of the solution to the given question; matrix multiplication is in general not communitative, so although from the given information AB=0 the object BA need not be the zero matrix. During the interview this serious error revealed in the student's written work was probed as follows:

R: Is this BA ? [*pointing to the first* BA *in the Step 2*]

S1: I left it for the last … wrote anything down.

Even after further probing, the student did not realize that in the context of matrices AB and BA represent different objects. With regard to Table 1 these imply the student was unable to correctly use mathematical symbolism in the context of matrices, firstly in its instrumental role and secondly as a communicative function. These, therefore, suggest that interactions among mental structures of actions, processes, objects and schema in the correct of matrices were not adequately connected. The implication here is that mathematical cognition was hindered since enculturation (Menary 2015) with regard to *mathematics as a logical symbolic system* (Godino 1996) was not achieved by this student.

CONCLUSION

The findings of this study cannot be generalized since there were only two participants. However, the framework arrived at to guide the examination of their written work served a useful purpose in that it provided a structure for the researcher to get an insight into those students' mathematical understanding, of linear algebra related concepts. That framework was used to make conclusions on the mathematical understanding of a student with regard to: (1) whether the understanding displayed was instrumental or relational in nature; (2) whether their use of mathematical symbolism served an instrumental role or communicative function; (3) the degree to which their conceptual system was logically organized in terms of a schema. The study indicated that the inability of a student to correctly use mathematical symbolism or to interpret mathematical symbolism coding linear algebra concepts, could lead to the student making incorrect conclusions. In particular it was found that one of the students could not interpret the equality of two matrices indicated in a symbolic form that she arrived at. It was also found for the question that checked whether students knew that in general matrix multiplication was not communicative, a student was unable to activate links among her knowledge schemata for the algebra structure $(a=b)^2$ and related information in the context $(A+b)^2$, for square matrices A and B. Here the student was unable to retrieve and reflect on knowledge represented in symbolic form, for example and are different objects in the context of matrices.

RECOMMENDATIONS

It is recommended that lecturers should have some sort of a framework to gauge the level of mathematical understanding of their students. The conclusions outlined indicate that the framework arrived at in Table 1 could suitably serve as such a framework or be modified, if required. Such a framework should be used to analyse the written work of students to gauge their level of mathematical understanding. It is recommended that such an examination of students' written work and relevant feedback to them could lead to an improvement in some aspects of their mathematical understanding, in particular the communicative function of mathematical symbolism which seems to be presently lacking. The researcher recommends that the framework to be used should be made available to students. Students should also be informed that the framework will be used to gauge their level of mathematical understanding with regard to the following: (1) whether their understanding displayed is instrumental or relational in nature; (2) whether their use of mathematical symbolism serves an instrumental role or communicative function; (3) the degree to which their conceptual system is logically organized in terms of a schema.

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